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Non-parallel flow effects on the stability of film flow down a right circular cone

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A global asymptotic solution for the linear stability of this flow with respect to axisymmetric disturbances is developed without invoking the usual quasi-parallel flow assumption. This solution predicts the occurrence of quasi-periodic spatially amplified disturbances relatively near the apex, which ultimately are stabilized further down the cone by the relative increase in viscous forces associated with the progressive thinning of the film. The wave speed and wavelength of these disturbances are found to decrease with increasing distance from the apex.

1. Introduction

Considerable progress has been made in describing both the linear and nonlinear stability of parallel film flow. Parallel film flow implies that the basic or unperturbed film flow has only one non-zero velocity component and a constant film thickness. However, no systematic solution has been developed to describe the stability of a non-parallel film flow. This is not surprising since even the linear stability problem for a non-parallel flow necessitates solving partial differential equations rather than ordinary differential equations as occur for parallel flows.

The stability of non-parallel flows has been analysed using three different approaches: the quasi-parallel flow approximation; the local expansion approach; and, the method of multiple scales (also referred to as the WKB, ray, or slowly varying flow approximation method). The quasi-parallel flow approximation greatly simplifies the stability problem for non-parallel flows by assuming that the stability problem is described by the equations appropriate to a parallel flow. However, the exact basic flow velocity profile is used in these parallel flow equations, thus introducing the dependence on the streamwise co-ordinate. This approach is deficient in that it can determine only the local rather than the global stability; furthermore, it does not take into account properly the effect of variation in the basic flow on the local growth.

Lanchon & Eckhaus (1964) developed a rational method of approximation for the stability of Blasius boundary layer flow. Their method, which has been referred to as local expansion theory, involves a double expansion in powers of the inverse Reynolds number about some arbitrary downstream point. As such, this approach accounts for the variation in the basic flow on the local growth rate, but cannot determine the

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complete solution as a function of the streamwise co-ordinate. Ling & Reynolds (1973) applied this method to the stability of the flat plate wake and two-dimensional jet as well as boundary-layer flow. However, these analyses have been shown to be incomplete. Gaster (1974) comments that these solutions do not account for the effect of the vertical structure in the ordering of the terms in the expansion procedure. Saric & Nayfeh (1975) note that these local expansion solutions have been developed only for temporally growing disturbances and thus do not properly compare with experiments involving spatially growing modes. Joseph (1974) has pointed out that the zeroth-order approximation of the solution of Ling & Reynolds (1973) for the two-dimensional jet, which reduces to the quasi-parallel flow solution, is not uniformly valid. Joseph presents a formal theory of bifurcation which allows for a certain flexibility in the choice of a zeroth-order approximation, thereby circumventing this problem encountered by nearly all the solutions advanced for the stability of non-parallel flows.

More recent developments in the theory of nearly parallel flows have used some variation of the method of multiple scales. Bouthier (1972, 1973) applied this method to the Blasius boundary layer. He showed that for non-parallel flows the concept of amplification or attenuation depends on the quantity considered; for example, the stream function may be growing, whereas the velocity components may be decaying at some point in the flow. This implies that the critical Reynolds number and neutral stability curve will depend on the flow quantity considered. Bouthier compared his predictions for the neutral stability curve based on the local kinetic energy, with the data of Ross *et al.* (1970) and obtained good agreement.

Gaster (1974) also applied the method of multiple scales to the Blasius boundary layer using a somewhat different formalism which accounted for the effect of the vertical structure in ordering the terms in the expansion procedure. He determined the neutral stability curves based on two integral measures of the kinetic energy and on the axial velocity evaluated at some selected position in the boundary layer. However, he obtained only qualitative agreement with the data of Schubauer & Skramstad (1948) and Ross *et al.* (1970).

Nayfeh, Mook & Saric (1974) developed a multiple scales expansion solution in rectangular co-ordinates, rather than similarity variables, for the Blasius boundary layer and for Falkner–Skan flows. Eagles & Weissman (1975) claim that the solution of Nayfeh *et al.* does not include the downstream variation of the eigenfunction in the expressions for the wavenumber and amplification rate. However, this effect may be insignificant at large Reynolds numbers since Nayfeh *et al.* obtain excellent agreement with the data of Schubauer & Skramstad and Ross *et al.* In this comparison they base their neutral stability curves on a zero spatial amplification factor normalized outside the boundary layer. Their choice of this flow quantity upon which to base their neutral stability curves accounts for why they obtained much better agreement with the available data than did Gaster (1974).

Eagles & Weissman (1975) have applied the method of multiple scales to describe the linear stability of slowly varying flow in a diverging straight-walled channel. They find the spatial growth rate to be a function of both the streamwise and cross-stream variables. Thus, the disturbances can pass successively through regions of growth and decay, however, far downstream all waves decay. They also comment that the growth or decay of the various flow quantities can be determined either absolutely or relative to the basic flow which is evolving as the disturbance grows.

The multiple scales method also can be applied to the stability of basic flows whose rate of change is slow in time rather than space. Hall & Parker (1976) have used this method in describing the linear stability of laminar flow in a suddenly blocked channel. They found that the quasi-steady flow approximation is a uniformly valid approximation for large Reynolds numbers.

The solutions discussed thus far have applied to flows characterized by large Reynolds numbers. DiPrima & Stuart (1972) applied the method of multiple scales to the low Reynolds number flow between eccentric rotating cylinders. Their non-parallel flow solution predicts an increase in the critical Taylor number over that predicted by the quasi-parallel flow approximation.

This brief review indicates that no systematic solutions have been developed for a non-parallel free surface flow for which the effects of surface tension and viscosity are significant. The analyses of Benjamin (1957) and Yih (1963) have established that parallel falling film flow is unstable with respect to long waves at very low Reynolds numbers. However, no attempt has been made to analyse the stability of a non-parallel film flow while invoking some rational method of approximation.

This paper analyses the linear stability of axisymmetric disturbances on film flow down a right circular cone. A solution developed for the basic non-parallel flow is reviewed in § 2. Section 3 develops a systematic solution to the associated linear stability problem. The theoretical predictions and conclusions are given in §§ 4 and 5, respectively. The interested reader desiring more details on this development is referred to Zollars (1974).

2. Basic flow solution

Figure 1 shows a sketch of rippled film flow down a right circular cone. The insert on the left in this figure gives an over-all view of this flow. The planar section on the right shows an axisymmetric disturbance of amplitude $\hat{h}(x,t)$ superimposed on a basic film flow of non-constant thickness $\bar{h}(x)$. An x axis is placed along the surface of cone such that $x = 0$ is at the apex; a y axis is placed such that $y = 0$ is at the surface of the cone; a radial co-ordinate measured from the axis of symmetry is given by

$$r = x\sin\beta + y\cos\beta$$

where β is the apex angle of the cone measured from the axis of symmetry.

An asymptotic solution for the basic film flow has been developed by Zollars & Krantz (1976) *via* a perturbation expansion in a small parameter δ which is a measure of the cross-stream to streamwise diffusion of vorticity. The resulting solution for the dimensionless stream function of the basic flow is given by

$$\begin{aligned}\bar{\psi} = {} & x\sin\beta\cos\beta\left(\frac{\bar{h}_0 y^2}{2}-\frac{y^3}{6}\right)+\delta\left[\frac{x\sin\beta\cos\beta h_1 y^2}{2}+\cos^2\beta\left(\frac{\bar{h}_0 y^3}{6}+\frac{\bar{h}_0^2 y^2}{4}-\frac{y^4}{12}\right)\right.\\ & +x\sin^2\beta(\bar{h}_0)_x\left(\frac{y^3}{6}-\frac{\bar{h}_0 y^2}{2}\right)+Re\sin\beta\cos^2\beta\left(\frac{\bar{h}_0 y^6}{180}-\frac{\bar{h}_0^2 y^5}{90}-\frac{y^7}{1260}+\frac{2\bar{h}_0^5 y^2}{45}\right)\\ & \left.+\frac{Re\,We\,\delta\cos\beta}{x}\left(\frac{\bar{h}_0 y^2}{2}-\frac{y^3}{6}\right)\right]+O(\delta^2), \qquad (2.1)\end{aligned}$$

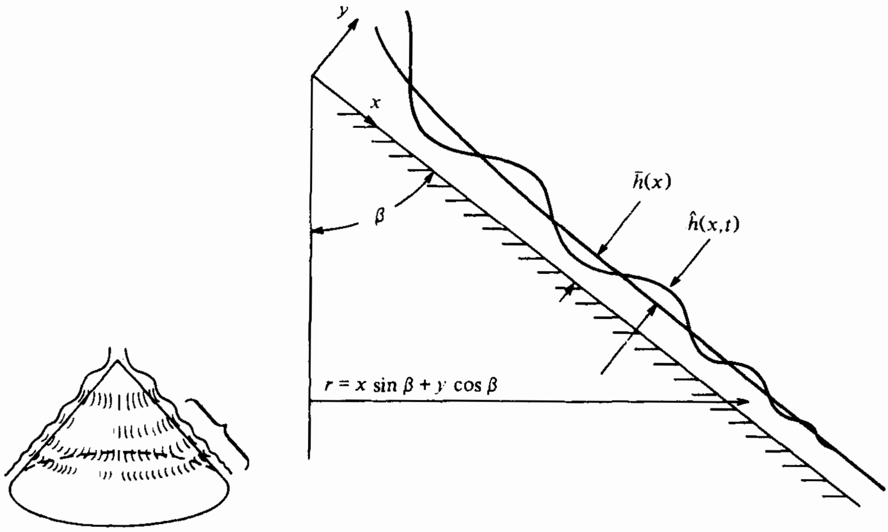

FIGURE 1. Geometry for rippled film flow down a right circular cone.

here $\bar{h}_0$ and $\bar{h}_1$ are the zeroth- and first-order terms respectively in the solution for $\bar{h}$ which is given in dimensionless form by

$$\bar{h} = Kx^{-\frac{1}{3}} - \delta\left(\tfrac{1}{9}\tan\beta K^2x^{-\frac{5}{3}} + \frac{K^2x^{-\frac{5}{3}}}{3\tan\beta} + \frac{4K^5x^{-\frac{8}{3}}}{105}\,Re\cos\beta + \frac{Re\,We\,\delta Kx^{-\frac{7}{3}}}{3\sin\beta}\right) + O(\delta^2), \quad (2.2)$$

where $K = 3/(\sin\beta\cos\beta)^{\frac{1}{3}}$. In arriving at the above equations the variables $\bar{\psi}$, x, y, and $\bar{h}$ were non-dimensionalized with the scale factors $\psi_c \equiv Q/2\pi L$, $x_c \equiv L$, $y_c = h_c \equiv (Q\nu/2\pi gL)^{\frac{1}{3}}$, respectively; Q is the volumetric flow rate, ν is the kinematic viscosity, g is the gravitational acceleration, and L is an unspecified length scale factor since there is no characteristic length in the streamwise direction. This non-dimensionalization introduces the Reynolds number $Re \equiv u_c y_c/\nu$, Weber number $We \equiv \sigma/\rho u_c^2 y_c$, and $\delta \equiv y_c/x_c$ in which $u_c \equiv \psi_c/x_c y_c$, σ is the surface tension, and ρ is the density. These scale factors are appropriate to a low-Reynolds-number flow in which the streamwise gravity forces are balanced by the principal viscous forces.

In arriving at (2.1) and (2.2) the following ordering arguments were made:

$$Re = O(1); \quad 1/r = O(1); \quad \text{and} \quad We = O(1/\delta).$$

These restrictions will be assessed quantitatively in § 4.

The zeroth-order terms in (2.1) and (2.2) correspond to the creeping flow solution. The first-order terms in δ give the first-order inertial and lateral curvature corrections. Although this perturbation scheme could be carried out to include higher-order terms in δ, this is not necessary for the purposes of this paper. The effect of the first-order terms is to decrease the film thickness and to increase the basic flow velocity predicted by the zeroth-order solution.

3. Perturbed flow solution

The equations of motion for an axisymmetric flow corresponding to the co-ordinate system shown in figure 1 are given by Millikan (1932). When appropriately non-dimensionalized they assume the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u\sin\beta+\delta v\cos\beta}{x\sin\beta+\delta y\cos\beta}=0; \tag{3.1}$$

$$Re\,\delta\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\delta\frac{\partial P}{\partial x}+\frac{\partial^2 u}{\partial y^2}-\delta^2\frac{\partial^2 v}{\partial x\,\partial y}+\left(\delta\frac{\partial u}{\partial y}-\delta^3\frac{\partial v}{\partial x}\right)\times\left(\frac{\cos\beta}{x\sin\beta+\delta y\cos\beta}\right)+\cos\beta; \tag{3.2}$$

$$Re\,\delta^2\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\delta^3\frac{\partial^2 v}{\partial x^2}-\delta\frac{\partial^2 u}{\partial x\,\partial y}+\left(\delta^3\frac{\partial v}{\partial x}-\delta\frac{\partial u}{\partial y}\right)\times\left(\frac{\sin\beta}{x\sin\beta+\delta y\cos\beta}\right)-\sin\beta. \tag{3.3}$$

The corresponding dimensionless boundary conditions are given by:

$$u=0 \quad \text{at} \quad y=0; \tag{3.4}$$

$$v=0 \quad \text{at} \quad y=0; \tag{3.5}$$

$$\left(\frac{\partial u}{\partial y}+\delta^2\frac{\partial v}{\partial x}\right)(1-\delta^2h_x^2)-2\delta^2h_x\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)=0 \quad \text{at} \quad y=h; \tag{3.6}$$

$$P-P_s-2\delta\frac{\partial v}{\partial y}+2\delta^3\frac{h_x^2}{1-h_x^2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+Re\,We\,\delta\left[\frac{\delta h_{xx}}{(1+\delta^2h_x^2)^{\frac{3}{2}}}-\frac{1-\delta h_x\tan\beta}{(x\tan\beta+\delta h)(1+\delta^2h_x^2)^{\frac{1}{2}}}\right]=0 \quad \text{at} \quad y=h. \tag{3.7}$$

The film thickness is related to the velocity components by the surface kinematic condition given in dimensionless form by

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}-v=0 \quad \text{at} \quad y=h. \tag{3.8}$$

In arriving at the above equations the streamwise and cross-stream velocity components u and v, pressure P, film thickness h, and independent variables x, y and t were non-dimensionalized with the scale factors $u_c \equiv (Q^2g/4\pi^2\nu L^2)^{\frac{1}{3}}$, $v_c \equiv Q/2\pi L^2$, $P_c \equiv (Q\rho^3g^2\nu/2\pi L)^{\frac{1}{3}}$, $h_c = y_c$, x_c, y_c, and $t_c \equiv x_c/u_c$ respectively, which were suggested by scaling the basic flow as described previously. The subscript x's, y's, and t's in these and subsequent equations denote differentiation with respect to x, y, or t respectively.

The boundary conditions given by (3.4) and (3.5) are the no-slip and no-flow conditions at the solid surface. Those given by (3.6) and (3.7) are the tangential and normal force balances at the free surface; the latter includes the streamwise and lateral curvature effects given by the terms in brackets. Note that we have not attempted to satisfy any boundary conditions for specified values of x. This proves to be no limitation since we will solve this system of equations *via* a perturbation expansion in δ. The resulting zeroth-order set of equations will not contain any derivatives with respect to x, and hence will not require any boundary conditions for this independent variable.

Since this is an axisymmetric flow it is convenient to define a dimensionless stream function ψ such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{3.9), (3.10}$$

A solution for the complete stream function ψ of the form $\psi = \bar{\psi}(x,y) + \hat{\psi}(x,y,t)$ will be sought, where $\bar{\psi}$ is given by (2.1), and $\hat{\psi}$ is the stream function of the perturbed flow. Similarly the overall film thickness is assumed to be of the form

$$h(x,t) = \bar{h}(x) + \hat{h}(x,t)$$

where $\bar{h}$ is given by (2.2). These equations can be substituted into (3.1)–(3.8); the resulting equations when linearized in the perturbation quantities assume the following form:

$$Re\,\delta\frac{\hat{\psi}_{yyt}}{r} + Re\left\{\delta\left[\left(\frac{1}{r}\right)\left(\frac{1}{r}\right)_x(2\bar{\psi}_{yy}\hat{\psi}_y + 2\bar{\psi}_y\hat{\psi}_{yy}) + \left(\frac{1}{r}\right)^2(\bar{\psi}_y\hat{\psi}_{xyy} + \bar{\psi}_{xyy}\hat{\psi}_y - \bar{\psi}_x\hat{\psi}_{yyy} - \bar{\psi}_{yyy}\hat{\psi}_x)\right]\right\} = 3\left(\frac{1}{r}\right)_y\hat{\psi}_{yyy} + \frac{\hat{\psi}_{yyyy}}{r} + \delta\left(\frac{1}{r}\right)^2\cos\beta\,\hat{\psi}_{yyy} + O(\delta^2), \tag{3.11}$$

$$\hat{\psi}_y = 0 \quad \text{at} \quad y = 0, \tag{3.12}$$

$$\hat{\psi}_x = 0 \quad \text{at} \quad y = 0, \tag{3.13}$$

$$\left(\frac{1}{r}\right)_y\hat{\psi}_y + \frac{\hat{\psi}_{yy}}{r} + \hat{h}\left[2\left(\frac{1}{r}\right)_y\bar{\psi}_{yy} + \frac{\bar{\psi}_{yyy}}{r}\right] + O(\delta^2) = 0 \quad \text{at} \quad y = \bar{h}, \tag{3.14}$$

$$3\hat{h}\left(\frac{1}{r}\right)_y\bar{\psi}_{yyy} + \hat{h}\frac{\bar{\psi}_{yyyy}}{r} + 2\left(\frac{1}{r}\right)_y\hat{\psi}_{yy} + \frac{\hat{\psi}_{yyy}}{r} - \delta\left\{Re\left[\left(\frac{1}{r}\right)\hat{\psi}_{yt} + 2\hat{h}\left(\frac{1}{r}\right)\left(\frac{1}{r}\right)_x\bar{\psi}_y\bar{\psi}_{yy} + \left(\frac{1}{r}\right)\left(\frac{1}{r}\right)_x\bar{\psi}_y\hat{\psi}_y - \hat{h}\left(\frac{1}{r}\right)^2\bar{\psi}_x\bar{\psi}_{yyy} + \hat{h}\left(\frac{1}{r}\right)^2\bar{\psi}_y\bar{\psi}_{xyy} + \left(\frac{1}{r}\right)^2\bar{\psi}_y\hat{\psi}_{xy} + \left(\frac{1}{r}\right)\left(\frac{1}{r}\right)_x\bar{\psi}_y\hat{\psi}_y + \left(\frac{1}{r}\right)^2\bar{\psi}_{xy}\hat{\psi}_y - \left(\frac{1}{r}\right)^2\bar{\psi}_x\hat{\psi}_{yy} - \left(\frac{1}{r}\right)^2\bar{\psi}_{yy}\hat{\psi}_x\right] - \cos\beta\left[\hat{h}\left(\frac{1}{r}\right)^2\bar{\psi}_{yyy} + \left(\frac{1}{r}\right)^2\hat{\psi}_{yy}\right] + \hat{h}_x\sin\beta\right\} + O(\delta^2) = 0 \quad \text{at} \quad y = \bar{h}, \tag{3.15}$$

$$\hat{h}_t + \hat{h}\left[\left(\frac{1}{r}\right)_y\bar{\psi}_y\bar{h}_x + \frac{\bar{\psi}_{yy}\bar{h}_x}{r} + \left(\frac{1}{r}\right)_y\bar{\psi}_x + \frac{\bar{\psi}_{xy}}{r}\right] + \frac{\bar{\psi}_y\hat{h}_x}{r} + \frac{\hat{\psi}_y\bar{h}_x}{r} + \frac{\hat{\psi}_x}{r} = 0 \quad \text{at} \quad y = \bar{h}. \tag{3.16}$$

The tangential and normal stress balances given by (3.14) and (3.15), and the kinematic surface condition given by (3.16) have been applied at the perturbed free surface by expanding these conditions in a Taylor series about the basic film thickness and linearizing the resulting equations in the perturbation quantities.

In arriving at (3.11) through (3.15), the same ordering arguments were used as in solving the basic flow problem. In particular, the ordering argument $1/r = O(1)$ restricts our solution from describing the flow in the immediate vicinity of the apex of the cone. Note that since $r = x\sin\beta + \delta y\cos\beta$, those terms in (3.11) through (3.16) containing derivatives of r with respect to y are $O(\delta)$ or smaller.

For the low Reynolds number thin film flows being considered here, the characteristic length for the diffusion of vorticity in the cross-stream direction will be small compared with that for the diffusion of vorticity in the streamwise direction. Therefore the dimensionless parameter δ will be small. Hence as was the case for the basic flow, a solution to (3.11) will be sought *via* a perturbation expansion in δ:

$$\hat{\psi} = \hat{\psi}_0 + \delta\hat{\psi}_1 + \delta^2\hat{\psi}_2 + \dots. \tag{3.17}$$

This method of solution was chosen rather than the method of multiple scales because it yields an analytical solution for the stability of long wavelength disturbances.

When (3.17) is substituted into (3.11)–(3.15) and only zeroth-order terms in δ are retained we obtain the following set of equations to be solved for $\hat{\psi}_0$:

$$(\hat{\psi}_0)_{yyyy} = 0, \tag{3.18}$$

$$(\hat{\psi}_0)_y = 0 \quad \text{at} \quad y = 0, \tag{3.19}$$

$$(\hat{\psi}_0)_x = 0 \quad \text{at} \quad y = 0, \tag{3.20}$$

$$(\hat{\psi}_0)_{yy} + \hat{h}(\overline{\psi}_0)_{yyy} = 0 \quad \text{at} \quad y = \overline{h}, \tag{3.21}$$

$$(\hat{\psi}_0)_{yyy} + \hat{h}(\overline{\psi}_0)_{yyyy} = 0 \quad \text{at} \quad y = \overline{h}. \tag{3.22}$$

When the first-order terms in δ are retained we obtain the following set of equations to be solved for $\hat{\psi}_1$:

$$(\hat{\psi}_1)_{yyyy} = \frac{2\cos\beta}{x\sin\beta}(\hat{\psi}_0)_{yyy} + Re\left\{(\hat{\psi}_0)_{yyt} - \frac{2}{x^2\sin\beta}[(\overline{\psi}_0)_{yy}(\hat{\psi}_0)_y + (\overline{\psi}_0)_y(\hat{\psi}_0)_{yy}]\right.$$
$$\left. + \frac{1}{x\sin\beta}[(\overline{\psi}_0)_y(\hat{\psi}_0)_{xyy} + (\overline{\psi}_0)_{xyy}(\hat{\psi}_0)_y - (\overline{\psi}_0)_x(\hat{\psi}_0)_{yyy} - (\overline{\psi}_0)_{yyy}(\hat{\psi}_0)_x]\right\}, \tag{3.23}$$

$$(\hat{\psi}_1)_y = 0 \quad \text{at} \quad y = 0, \tag{3.24}$$

$$(\hat{\psi}_1)_x = 0 \quad \text{at} \quad y = 0, \tag{3.25}$$

$$(\hat{\psi}_1)_{yy} - \frac{\cos\beta}{x\sin\beta}(\hat{\psi}_0)_y - \frac{y\cos\beta}{x\sin\beta}(\hat{\psi}_0)_{yy} + \hat{h}\left[(\overline{\psi}_1)_{yyy} - \frac{2\cos\beta}{x\sin\beta}(\overline{\psi}_0)_{yy}\right.$$
$$\left. - \frac{y\cos\beta}{x\sin\beta}(\overline{\psi}_0)_{yyy}\right] = 0 \quad \text{at} \quad y = \overline{h}, \tag{3.26}$$

$$(\hat{\psi}_1)_{yyy} - \frac{y\cos\beta}{x\sin\beta}(\hat{\psi}_0)_{yyy} - \frac{2\cos\beta}{x\sin\beta}(\hat{\psi}_0)_{yy} + \hat{h}\left[(\overline{\psi}_1)_{yyyy} - \frac{y\cos\beta}{x\sin\beta}(\overline{\psi}_0)_{yyyy}\right.$$
$$\left. - \frac{3\cos\beta}{x\sin\beta}(\overline{\psi}_0)_{yyy}\right] + Re\left\{-(\hat{\psi}_0)_{yt} + \frac{2(\overline{\psi}_0)_y(\hat{\psi}_0)_y}{x^2\sin\beta} - \frac{(\overline{\psi}_0)_y(\hat{\psi}_0)_{xy}}{x\sin\beta} - \frac{(\overline{\psi}_0)_{xy}(\hat{\psi}_0)_y}{x\sin\beta}\right.$$
$$\left. + \frac{(\overline{\psi}_0)_x(\hat{\psi}_0)_{yy}}{x\sin\beta} + \frac{(\overline{\psi}_0)_{yy}(\hat{\psi}_0)_x}{x\sin\beta} + \hat{h}\left[\frac{2(\overline{\psi}_0)_y(\overline{\psi}_0)_{yy}}{x^2\sin\beta} + \frac{(\overline{\psi}_0)_x(\overline{\psi}_0)_{yyy}}{x\sin\beta} - \frac{(\overline{\psi}_0)_y(\overline{\psi}_0)_{xyy}}{x\sin\beta}\right]\right\}$$
$$+ \cos\beta\left[\frac{\hat{h}(\overline{\psi}_0)_{yyy}}{x\sin\beta} + \frac{(\hat{\psi}_0)_{yy}}{x\sin\beta}\right] - \hat{h}_x x\sin^2\beta = 0 \quad \text{at} \quad y = \overline{h}. \tag{3.27}$$

Note that the streamwise and lateral curvature of the perturbed flow do not appear in (3.18) through (3.27) since these effects will appear only when higher-order terms in δ are retained. However, the lateral curvature effect of the basic flow will be introduced through the terms in $\bar{\psi}_1$ and its derivatives.

The perturbation expansion scheme employed here greatly simplifies the solution of (3.11) because neither (3.18) for $\hat{\psi}_0$ nor (3.23) for $\hat{\psi}_1$ contains derivatives of these dependent variables with respect to x. The x dependence of $\hat{\psi}_0$ enters only through the tangential and normal stress boundary conditions given by (3.21) and (3.22), which are evaluated at $\bar{h}$.

The solution of (3.18) through (3.27) is straightforward and given by

$$\begin{aligned}\hat{\psi} = {} & \hat{h}\, x\sin\beta\cos\beta\frac{y^2}{2} + \delta\left\{Re\, x\sin\beta\cos^2\beta\left[\frac{\hat{h}}{x}\left(\frac{y^6}{180}-\frac{\bar{h}_0 y^5}{60}-\frac{\bar{h}_0^2 y^4}{24}\right.\right.\right.\\ & \left.+\frac{\bar{h}_0^3 y^3}{18}+\frac{\bar{h}_0^4 y^2}{4}\right)+\hat{h}(\bar{h}_0)_x\left(\frac{y^5}{120}-\frac{\bar{h}_0 y^4}{12}+\frac{\bar{h}_0^2 y^3}{6}-\frac{\bar{h}_0^3 y^2}{12}\right)\\ & \left.+\hat{h}_x\left(\frac{\bar{h}_0 y^5}{120}-\frac{\bar{h}_0^2 y^4}{24}+\frac{\bar{h}_0^4 y^2}{6}\right)\right]+\hat{h}\cos^2\beta\left(\frac{y^3}{6}+\frac{\bar{h}_0 y^2}{2}\right)\\ & \left.+x\sin^2\beta\left[\frac{\hat{h}_x y^3}{6}-\frac{\hat{h}_x\bar{h}_0 y^2}{2}-\frac{\hat{h}(\bar{h}_0)_x y^2}{2}\right]+\frac{Re\, We\,\delta\cos\beta\hat{h}y^2}{2x}\right\}\\ & +O(\delta^2).\end{aligned} \tag{3.28}$$

The above can be substituted into the kinematic surface condition given by (3.16) to obtain the following differential equation to be solved for the disturbance amplitude $\hat{h}$:

$$\begin{aligned}\hat{h}_t = {} & -\hat{h}\left[\frac{\cos\beta K^2 x^{-\frac{5}{3}}}{3}+\delta\left(\frac{-3\cos\beta x^{-3}}{\sin^2\beta}-\frac{x^{-3}}{3\cos\beta}-\frac{148\, Re\, x^{-4}}{35\sin^2\beta}\;\frac{-5\, Re\, We\,\delta\cos\beta K^2 x^{-\frac{11}{3}}}{9\sin\beta}\right)\right]\\ & -\hat{h}_x\left[\cos\beta K^2 x^{-\frac{2}{3}}+\delta\left(-\frac{\cos\beta x^{-2}}{2\sin^2\beta}+\frac{x^{-2}}{3\cos\beta}-\frac{3\, Re\, x^{-3}}{35\sin^2\beta}+\frac{Re\, We\,\delta\cos\beta K^2 x^{-\frac{8}{3}}}{3\sin\beta}\right)\right]\\ & -\hat{h}_{xx}\,\delta\left(\frac{6\, Re\, x^{-2}}{5\sin^2\beta}-\frac{1}{x\cos\beta}\right).\end{aligned} \tag{3.29}$$

The above admits a solution of the form $\hat{h}(x,t) = X(x)\,.\,T(t)$. Allowing the separation constant to be either real or complex results in a solution for disturbances which grow both spatially and temporally. Hence, since disturbances in most film flows grow spatially the separation constant must be imaginary. It is interesting to note that although (3.29) admits purely spatially growing disturbances, it does not admit purely temporally growing disturbances.

The solution for $T(t)$ is found to be

$$T = ke^{\pm ik_s t}, \tag{3.30}$$

where ik_s is the separation constant and k is an integration constant. From the above we see that $k_s \equiv \omega$ the angular frequency. Hence the solution for $X(x)$ is obtained from

$$\begin{aligned}\mp i\omega = {} & \frac{1}{X}\frac{\partial^2 X}{\partial x^2}\,\delta\left(\frac{6\, Re\, x^{-2}}{5\sin^2\beta}-\frac{1}{x\cos\beta}\right)\\ & +\frac{1}{X}\frac{\partial X}{\partial x}\left[\cos\beta K^2 x^{-\frac{2}{3}}+\delta\left(\frac{-\cos\beta x^{-2}}{2\sin^2\beta}+\frac{x^{-2}}{3\cos\beta}-\frac{3\, Re\, x^{-3}}{35\sin^2\beta}+\frac{Re\, We\,\delta\cos\beta K^2 x^{-\frac{8}{3}}}{3\sin\beta}\right)\right]\end{aligned}$$

$$+\left[\frac{\cos\beta K^2x^{-\frac{5}{3}}}{3}+\delta\left(\frac{3\cos\beta x^{-3}}{\sin^2\beta}-\frac{x^{-3}}{3\cos\beta}-\frac{148\,Re\,x^{-4}}{35\sin^2\beta}-\frac{5\,Re\,We\,\delta\cos\beta K^2x^{-\frac{11}{3}}}{9\sin\beta}\right)\right]. \tag{3.31}$$

Seeking a solution to the above of the form $X(x) = \exp[f(x)+\delta g(x)+O(\delta^2)]$ and retaining terms through first-order in δ yields

$$X = \frac{k'}{x^{\frac{1}{3}}}\exp\left\{\mp i\omega\left[\frac{3x^{\frac{5}{3}}}{5K^2\cos\beta}+\delta\left(\frac{3x^{\frac{1}{3}}}{2K^4\sin^2\beta\cos\beta}-\frac{x^{\frac{1}{3}}}{K^4\cos^3\beta}\right.\right.\right.$$
$$\left.\left.-\frac{9\,Re\,x^{-\frac{2}{3}}}{70K^4\sin^2\beta\cos^2\beta}+\frac{Re\,We\,\delta x^{-\frac{1}{3}}}{K^2\sin\beta\cos\beta}\right)\right]+\delta\left(\frac{2\omega^2\,Re\,x}{15}-\frac{\omega^2x^2}{2K^6\cos^4\beta}\right.$$
$$\left.\left.-\frac{2x^{-\frac{4}{3}}}{3K^2\cos^2\beta}-\frac{17x^{-\frac{4}{3}}}{8K^2\sin^2\beta}-\frac{11\,Re\,x^{-\frac{7}{3}}}{7K^2\sin^2\beta\cos\beta}-\frac{Re\,We\,\delta x^{-2}}{3\sin\beta}\right)\right\}, \tag{3.32}$$

where k' is an integration constant.

Equations (3.30) and (3.32) can be combined to obtain the general solution for $\hat{h}(x,t)$. Recasting the exponentials corresponding to positive and negative values of the separation constant in terms of trigonometric functions then yields the following equation for $\hat{h}(x,t)$:

$$\hat{h} = k'_c\frac{e^{\delta B'x}}{x^{\frac{1}{3}}}\cos[\omega(A'x-t)] \tag{3.33}$$

where

$$A' = \frac{3x^{\frac{2}{3}}}{5K^2\cos\beta}+\delta\left(\frac{x^{-\frac{2}{3}}}{2K\sin\beta}-\frac{\sin\beta x^{-\frac{2}{3}}}{3K\cos^2\beta}-\frac{3\,Re\,x^{-\frac{5}{3}}}{70K\sin\beta\cos\beta}+\frac{Re\,We\,\delta x^{-\frac{4}{3}}}{K^2\sin\beta\cos\beta}\right) \tag{3.34}$$

$$B' = \frac{2w^2\,Re}{15}-\frac{\omega^2\sin^2\beta x}{18\cos^2\beta}-\frac{2x^{-\frac{7}{3}}}{3K^2\cos^2\beta}-\frac{17x^{-\frac{7}{3}}}{8K^2\sin^2\beta}-\frac{11\,Re\,x^{-\frac{10}{3}}}{7K^2\sin^2\beta\cos\beta}$$
$$-\frac{Re\,We\,\delta x^{-3}}{3\sin\beta}. \tag{3.35}$$

If (3.33) is recast in dimensional form, the unspecified scale factor L will necessarily cancel out. For purposes of generalizing these results and presenting them in useful graphical form, it is convenient to define a new length scale factor $(Q\nu/2\pi g)^{\frac{1}{3}}$ and a new time scale factor $(2\pi\nu^3/Qg^3)^{\frac{1}{3}}$. These new scale factors are suggested by the dimensional form of (3.33) which then assumes the following form in terms of the redefined dimensionless variables h_*, x_*, and t_*:

$$\hat{h}_* = k_c\frac{e^{Bx_*}}{x_*^{\frac{1}{3}}}\cos A[x_*-(A_1/A)t_*], \tag{3.36}$$

where k_c is an unspecified integration constant equal to the dimensional initial amplitude of the infinitesimal disturbance, and

$$A = A_1\left(\frac{3x_*^{\frac{2}{3}}}{5K^2\cos\beta}+\frac{x_*^{-\frac{2}{3}}}{2K\sin\beta}-\frac{\sin\beta x_*^{-\frac{2}{3}}}{3K\cos^2\beta}-A_2^{\frac{3}{4}}\frac{3x_*^{-\frac{5}{3}}}{70K\sin\beta\cos\beta}\right.$$
$$\left.+A_2^{-\frac{1}{2}}A_3\frac{x_*^{-\frac{4}{3}}}{K^2\sin\beta\cos\beta}\right) \tag{3.37}$$

$$B = A_1^2\left(\tfrac{2}{15}A_2^{\frac{3}{4}} - \frac{x_*\sin^2\beta}{18\cos^2\beta}\right) - \left(\frac{2x_*^{-\frac{7}{3}}}{3K^2\cos^2\beta} + \frac{17x_*^{-\frac{7}{3}}}{8K^2\sin^2\beta} + A_2^{\frac{3}{4}}\frac{11x_*^{-\frac{10}{3}}}{7K^2\sin^2\beta\cos\beta} + A_2^{-\frac{1}{3}}A_3\frac{x_*^{-3}}{3\sin\beta}\right) \quad (3.38)$$

and $A_1 \equiv \omega(2\pi\nu^3/Qg^3)^{\frac{1}{4}}$; $A_2 \equiv Qg^{\frac{1}{3}}/(2\pi\nu^{\frac{5}{3}})$; and $A_3 \equiv \sigma/(\rho\nu^{\frac{4}{3}}g^{\frac{1}{3}})$. This choice of dimensionless groups is convenient since A_1 is a dimensionless disturbance frequency; A_2 characterizes the dynamics of the flow; and A_3 depends only on the fluid properties. Note that (3.36) indicates that infinitesimal disturbances on this flow will be quasi-periodic with dimensionless wavenumber A and dimensionless wave speed A_1/A, both of which depend on the axial distance.

4. Discussion

This analysis for non-parallel film flow down a cone indicates that a disturbance of frequency ω has a wave number and wave speed which are functions of the streamwise co-ordinate, and a non-exponential spatial amplification factor.

Equation (3.36) implies that this flow is stable since h_* becomes zero for sufficiently large x_*. However, this does not imply that all disturbances decay everywhere in this flow. Indeed, (3.36) indicates that disturbances can be amplified locally both in an absolute and a relative sense. That is, if $d[\exp(Bx_*)/x_*^{\frac{1}{3}}]/dx_* > 0$ a disturbance of frequency ω will experience an absolute increase in amplitude. However, if $d[\exp(Bx_*)/\bar{h}_* x_*^{\frac{1}{3}}]/dx_* > 0$ a disturbance of frequency ω will increase in amplitude relative to the basic film thickness $\bar{h}_*$ which decreases monotonically with distance from the apex of the cone. These two manifestations of local amplification will be referred to as absolute and relative growth, respectively. A similar distinction has been made by Eagles & Weissman (1975) in discussing the linear stability of slowly varying flow in a diverging channel.

It is convenient to define a modified spatial amplification factor for absolute growth as follows:

$$G_a \equiv B_{x*}x_* + B - 1/(3x_*). \quad (4.1)$$

This modified spatial amplification factor has the property that if $G_a > 0$ the disturbance will be amplified. The neutral growth curve for absolute growth is defined by $G_a = 0$.

Substituting (3.38) into (4.1) yields

$$G_a = A_1^2\Big(\underbrace{\tfrac{2}{15}A_2^{\frac{3}{4}}}_{(a)} - \underbrace{\frac{x_*\sin^2\beta}{9\cos^2\beta}}_{(b)}\Big) + \underbrace{\left(\frac{8x_*^{-\frac{7}{3}}}{9K^2\cos^2\beta} + \frac{17x_*^{-\frac{7}{3}}}{6K^2\sin^2\beta} + A_2^{\frac{3}{4}}\frac{11x_*^{-\frac{10}{3}}}{3K^2\sin^2\beta\cos\beta} + A_2^{-\frac{1}{3}}A_3\frac{2x_*^{-3}}{3\sin\beta}\right)}_{(c)} - \underbrace{\frac{1}{3x_*}}_{(d)}. \quad (4.2)$$

All terms in (4.2) arise from the first-order terms in δ except for term (d) which is a stabilizing effect arising from the zeroth-order thinning of the basic flow. Hence the zeroth-order solution to this stability problem predicts that all disturbances will decay everywhere in the flow; one must retain at least first-order terms in δ to ascertain the

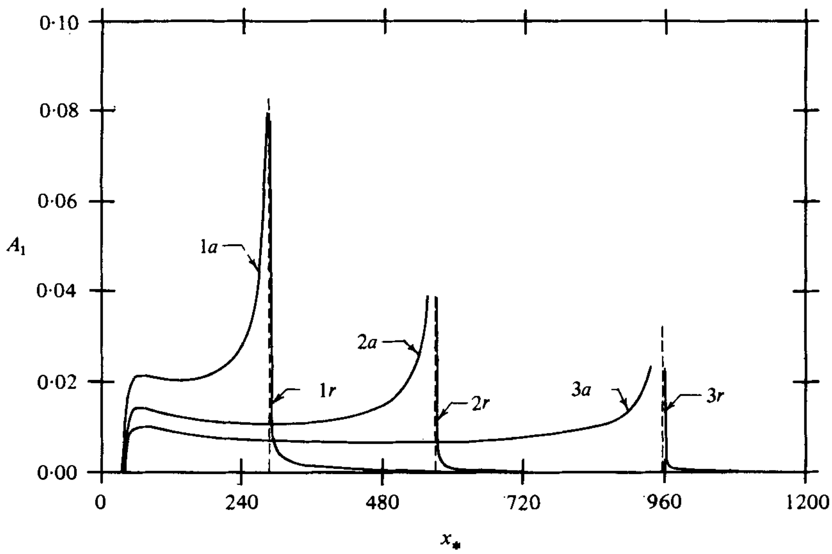

FIGURE 2. Neutral growth curves for $A_3 = 3388$ and $\beta = 30^\circ$. *a*, absolute neutral growth; *r*, relative neutral growth. 1, $A_2 = 340{\cdot}6$; 2, $A_2 = 851{\cdot}5$; 3, $A_2 = 1703$.

nature of the local amplification. Term (*a*) in (4.2) is a destabilizing effect arising from the inertial transfer terms entering at first-order in the equations for the perturbed flow. Term (*b*) is the first-order stabilizing effect of viscosity. The four terms denoted by (*c*) are first-order destabilizing effects arising from the increase in inertial transfer due to the increase in basic flow velocity associated with the first-order thinning effects in the basic flow film thickness.

Similarly, a modified spatial amplification factor for relative growth can be defined as follows:

$$G_r \equiv (B_{x_*} x_* + B)\bar{h}_* - \bar{h}_*/(3x_*) - \bar{h}_{x_*}. \tag{4.3}$$

This has the property that if $G_r > 0$ the disturbance will grow relative to the basic film thickness. The neutral growth curve for relative growth is defined by $G_r = 0$.

Substituting (2.2) and (3.38) into (4.3) yields

$$G_r = A_1^2[\underbrace{(\tfrac{2}{15})A_2^{\frac{3}{2}}}_{(a)} - \underbrace{(\tfrac{1}{9})x_*\tan^2\beta]}_{(b)} + \underbrace{x_*^{-\frac{7}{3}}[(\tfrac{4}{27})K\tan\beta + \tfrac{1}{2}K/\tan\beta] + x_*^{-\frac{10}{3}}A_2(\tfrac{301}{315})K/\sin\beta}_{(c)}. \tag{4.4}$$

Terms (*a*), (*b*), and (*c*) appearing in the above have the same physical interpretation as do these terms appearing in (4.2). Note however, that term (*c*) no longer includes any surface tension effects, and that term (*d*) in (4.2) does not appear in (4.4).

Figure 2 shows a plot of the dimensionless neutral growth frequency A_1 *versus* dimensionless axial distance x_* for a water film ($A_3 = 3388$) flowing down a 30° cone at dimensionless flow rates $A_2 = 340{\cdot}6$, $851{\cdot}5$, and 1703, corresponding to the dimensional flow rates 0·1, 0·25, and 0·5 $cm^3\,s^{-1}$. The three solid lines denoted by 1*a*, 2*a*, and 3*a* in this figure are the loci of those dimensionless frequencies for which $G_a = 0$. The region above each of these solid lines corresponds to frequencies which are amplified with respect to absolute growth; i.e. those for which $G_a > 0$ at that value of A_2.

The region below corresponds to frequencies which decay with respect to absolute growth. Note that each of these curves is associated with an upstream point along the surface of the cone above which all frequencies are amplified. This point on the neutral growth curve is determined when term (*c*) is equal to term (*d*) in (4.2). For the three flow rates shown in figure 2 this upstream terminus of the neutral stability curve is given by 0·681 cm at $A_2 = 340{\cdot}6$; 0·891 cm at $A_2 = 851{\cdot}5$; and 1·18 cm at $A_2 = 1703$. Note also that each neutral growth curve for absolute growth appears to approach infinite frequency asymptotically at some downstream position. This position, denoted by the dashed vertical line, is determined when term (*a*) is equal to term (*b*) in (4.2). Beyond this position all frequencies decay with respect to absolute growth. For the three flow rates shown in figure 2 this downstream terminus of the neutral growth curve is given by 5·73 cm at $A_2 = 340{\cdot}6$; 14·3 cm at $A_2 = 851{\cdot}5$; and 28·7 cm at $A_2 = 1703$. Note that these three neutral growth curves are terminated at some higher frequency which is seen to decrease as the dimensionless flow rate A_2 increases. This upper frequency bound arises from our ordering argument $Re = O(1)$. This ordering argument was converted into an inequality by assuming it would be satisfied if $Re \leqslant 0{\cdot}1/\delta$. The parameter δ contains the unspecified characteristic length L which was chosen to be the local wavelength, by analogy with parallel film flows. Hence for a given fluid and specified volumetric flow rate the upper frequency bound is associated with that value of x_* at which the local wavelength for neutral growth satisfies the equality in the inequality above. It would appear that the ordering argument $1/r = O(1)$ would determine an upstream bound on the neutral growth curve. However, this bound occurs somewhat upstream of the point above which all disturbances are amplified; hence, it does not prove to be limiting in practice.

The three solid lines denoted by 1*r*, 2*r*, and 3*r* in figure 2 are the loci of those dimensionless frequencies for which $G_r = 0$. For these curves the region below each solid line corresponds to frequencies which are amplified with respect to relative growth; i.e. for which $G_r > 0$ at that value of A_2. The region above each solid line corresponds to frequencies which decay with respect to relative growth. The neutral growth curves for relative growth are terminated at an upper frequency bound which is determined again by our ordering argument $Re = O(1)$. Equation (4.4) indicates that the neutral growth curves for relative growth asymptotically approach the line $A_1 = 0$ as $x_* \to \infty$. Thus there are some frequencies which are amplified in the relative growth sense at all positions along the surface of the cone.

A physical interpretation of these neutral growth curves for both absolute and relative growth can be obtained by considering the magnitude of the various terms in (4.2) and (4.4). At small values of x_* the sum of terms (*a*) and (*c*) is greater than that of terms (*b*) and (*d*); hence all disturbances are amplified with respect to both absolute and relative growth. As x_* increases we reach a point at which term (*c*) becomes equal to term (*d*); hence zero frequency disturbances become neutrally amplified with respect to absolute growth. As x_* increases further term (*c*) becomes insignificant and term (*b*) becomes of increasingly greater importance, progressively stabilizing higher frequencies with respect to absolute growth. Eventually at some x_* terms (*a*) and (*b*) become equal and all disturbances are stabilized with respect to absolute growth. Hence, in the case of figure 2 the inertial transfer from the basic flow velocity due to the disturbances is enhanced near the apex by the increase in basic flow velocity due to the first-order thinning effects. This destabilization is counteracted in part near

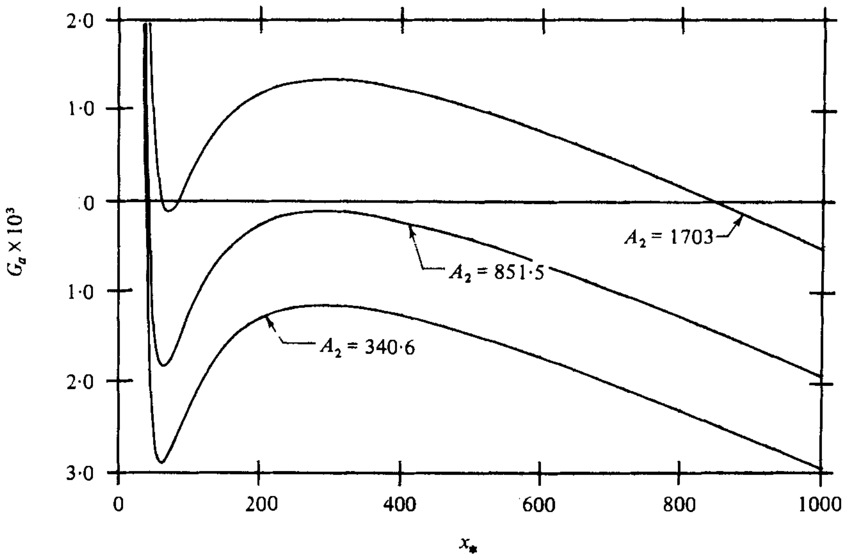

FIGURE 3. Modified spatial amplification factor for absolute growth, $A_1 = 0{\cdot}01$, $A_3 = 3388$, and $\beta = 30°$.

the apex by the zeroth-order thinning of the basic flow. However, the principal stabilizing effect further downstream is contained in term (*b*) which represents the action of increasing viscous forces as the film progressively thins. Note that to the left of the dashed vertical line in figure 2 all disturbances are unstable with respect to relative growth since the stabilizing term (*d*) does not appear in (4.4). To the right of the vertical line term (*b*) is greater than term (*a*); hence as x_* increases the destabilizing term (*c*) becomes progressively less significant. Thus, the flow is stabilized with respect to relative growth at progressively lower frequencies.

The interplay of these various stabilizing and destabilizing factors can give rise to an interesting possibility as indicated by the slight local maxima in the three neutral growth curves for absolute growth in figure 2. For a specified flow rate of a given fluid. it is possible for a small band of frequencies to be amplified near the apex of the cone; decay somewhat further down the cone; be amplified again yet further down the cone; and ultimately decay on the remainder of the cone. This possibility is demonstrated more clearly in figure 3 which shows a plot of G_a as a function of x_* for a disturbance having a dimensionless frequency $A_1 = 0{\cdot}01$; the cone angle, fluid, and dimensionless flow rates in figure 3 are the same as those of figure 2. Note that for the curve corresponding to $A_2 = 1703$ it is possible for the spatial amplification factor of this disturbance to change sign three times as the disturbance progresses down the cone. An increase in the dimensionless flow rate A_2 is seen to increase the spatial amplification factor.

Equation (3.36) indicates that the dimensionless wave speed, given by $c_r = A_1/A$, is independent of the dimensionless frequency at the order in δ which the stability problem has been solved here. Except for a relatively small region near the apex, the dimensionless wave speed is given by

$$c_r \simeq \frac{15}{(\tan\beta \sin\beta)^{\frac{1}{3}} x_*^{\frac{2}{3}}} \tag{4.5}$$

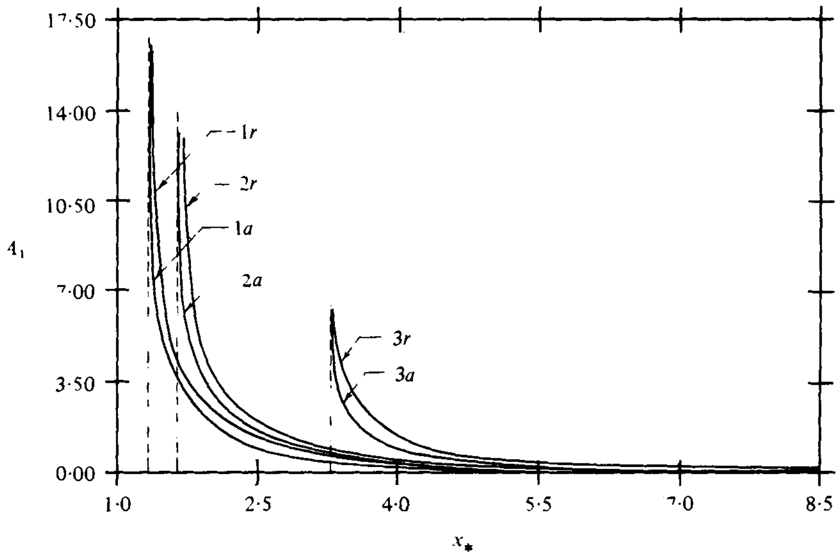

FIGURE 4. Neutral growth curves for $A_3 = 1{\cdot}78$ and $\beta = 60°$. *a*, absolute neutral growth; *r*, relative neutral growth. 1, $A_2 = 4{\cdot}964$; 2, $A_2 = 6{\cdot}619$; 3, $A_2 = 16{\cdot}55$.

for $x_* \gg 1$. The corresponding local dimensionless wavelength is given by $\lambda = 2\pi/A = 2\pi c_r/A_1$. Hence the local wave speed and wavelength decrease as a disturbance of fixed frequency progresses down the cone.

Figure 2 indicates that an increase in dimensionless flow rate destabilizes the flow as one would expect. Parameter studies indicated that an increase in cone angle β has a stabilizing effect on the flow as does a decrease in the dimensionless fluid properties group A_3.

The neutral growth curves do not always appear as those in figure 2. If one has highly stabilizing flow corresponding to a large cone angle β, small fluid properties group A_3, and small dimensionless flow rate A_2, the neutral growth curves may appear as those in figure 4. This figure is for a fluid having a properties group $A_3 = 1{\cdot}78$, corresponding to a light mineral oil ($\rho = 0{\cdot}868\,\mathrm{g\,cm^{-3}}$; $\nu = 1{\cdot}686\,\mathrm{cm\,s^{-1}}$; and $\sigma = 30{\cdot}8\,\mathrm{dyn\,cm^{-1}}$ at 25 °C) flowing down a 60° cone. The three dimensionless flow rates $A_2 = 4{\cdot}964$, $6{\cdot}619$, and $16{\cdot}55$ correspond to dimensional flow rates of 7·5, 10·0, and 25·0 $\mathrm{cm^3\,s^{-1}}$ respectively. In this case all disturbances below the neutral growth curves are amplified, whereas all disturbances above decay. Again the neutral growth curves for absolute growth are denoted by 1*a*, 2*a*, and 3*a*, and those for relative growth by 1*r*, 2*r*, and 3*r*. The physical interpretation of these neutral growth curves again can be obtained by considering the terms in (4.2) and (4.4). To the left of the vertical asymptote denoted by the dashed line, the sum of the destabilizing terms (*a*) and (*c*) is greater than the sum of the stabilizing terms (*b*) and (*d*) for all frequencies. At the vertical asymptote term (*a*) is equal to term (*b*); however, this occurs at a point sufficiently close to the apex so that term (*c*) is still of importance. Hence term (*c*) can still effect the destabilization of lower frequency disturbances at somewhat larger values of x_*. As x_* continues to increase term (*b*) progressively increases and stabilizes lower frequencies as term (*c*) becomes less significant. Hence for highly stabilizing flow conditions the two destabilizing effects are significant only very near the apex. Ultimately in the case of absolutely growing disturbances terms (*c*) and (*d*) in (4.2) become equal; this determines a downstream point along the surface of the cone beyond which all disturbances decay with respect to the absolute growth. For the three flow rates shown in figure 4, the dimensional downstream bound on the neutral growth curves for absolute growth is given by 1·29 cm at $A_2 = 4{\cdot}964$; 1·45 cm at $A_2 = 6{\cdot}619$; and 2·19 cm at $A_2 = 16{\cdot}55$. However, since term (*d*) is not contained in (4.4), there is always a small band of long wavelength disturbances which are amplified with respect to relative growth.

Notice that the long wave solution obtained here for film flow down a right circular cone does not reduce to the planar film flow solutions of Benjamin (1957) and Yih (1963) in the limit of $\beta = 0$. The ordering argument $1/r = O(1)$ precludes considering this limit process since the first term in the expansion of $1/r$ for small δ is $1/(x \sin \beta)$. Furthermore, the limit $\beta = 0$ would not yield the planar film flow solution since the lateral curvature effect is still retained in this limit. This limit then would correspond to the flow of a cylindrical film whose basic flow has a constant film thickness. An asymptotic long wave solution for the linear stability for the latter flow has been developed by Krantz & Zollars (1976).

It is of interest to compare the results obtained here for the stability of non-parallel film flow to those obtained by Benjamin (1957) and Yih (1963) for planar film flow and by Lin & Liu (1975) and Krantz & Zollars (1976) for cylindrical film flow.

The present results indicate that under some conditions, such as those of figure 2, all frequencies above the neutral growth curve for absolute growth are amplified; under other conditions, such as those of figure 4, all frequencies below the neutral growth curve for absolute growth decay. Which of these situations will prevail, depends upon the magnitude of the viscous stabilizing term associated with the film thinning, relative to the destabilizing terms in (4.2). In contrast, parallel film flow analyses indicate that unstable frequencies always occur below the neutral stability curves for the surface of 'soft' (as opposed to shear) instabilities being considered here. Furthermore an increase in surface tension, which is reflected in an increase in the properties group A_3, is found to have a destabilizing influence on non-parallel film flow, but a stabilizing effect on parallel film flow.

These differences between the non-parallel and parallel film flow predictions arise in part from the surface tension effects which are neglected in the present analysis. When the ordering argument $We = O(1/\delta)$ is invoked, the only effect of surface tension included to terms of first order in δ is that of the lateral curvature of the basic flow. This term accounts for the destabilizing influence of increasing surface tension; its effect is to thin the basic flow thereby increasing the basic flow velocity and associated inertial transfer terms in the equations for the perturbed flow. Krantz & Zollars also found the lateral curvature effect to be destabilizing for cylindrical film flow. The neglected surface tension effects include the streamwise curvature of the basic flow, and both the lateral and streamwise curvatures of the perturbed flow. Only the latter surface tension effect is relevant in the planar film analyses of Benjamin and Yih. The streamwise curvature of the perturbed flow stabilizes the flow to higher frequency short waves which have large curvature. This surface tension effect is the principal stabilizing effect in parallel film flow analyses and accounts for the stabilization of the higher frequency disturbances. The principal stabilizing influence in the present non-parallel film flow analysis is the thinning of the basic flow.

The neglected surface tension effect could be accounted for in the present analysis if second-order terms in δ were retained. However, it is not feasible to carry out this perturbation solution analytically to higher-order terms owing to the extremely tedious algebra involved. Alternately, these neglected surface tension effects would be included with first-order terms in δ if the ordering argument $We = O(1/\delta^2)$ were invoked. Unfortunately this ordering argument does not permit an analytical solution to the perturbed flow equations even at zeroth order.

It is interesting to speculate as to the influence of these neglected surface-tension effects. The streamwise curvature of the perturbed flow clearly will stabilize the flow to higher frequencies, thus resulting in an upper branch of the neutral growth curve for absolute growth. Thus, only a finite band of frequencies will be amplified at any streamwise location. This effect will become progressively more important further down the cone since the wavelength of a given frequency decreases with streamwise distance. The lateral curvature of the perturbed flow will be a destabilizing effect, since the troughs of the waves have smaller radii of curvature than do the crests; the latter results in a capillary pressure force which induces flow from the troughs into the crests. It is difficult to conjecture as to the effect of the streamwise curvature of the basic flow; its influence should be significant only at the apex of the cone where the streamwise curvature is appreciable.

5. Conclusions

An asymptotic long wave solution has been developed for the linear stability of axisymmetric disturbances on non-parallel film flow down a right circular cone. Spatially amplified, quasi-periodic wave forms are predicted.

This solution indicates that although this flow is stable, some disturbances will always be amplified at least near the apex of the cone. These disturbances ultimately will be stabilized further down the cone owing to the decrease in local Reynolds number associated with the progressive thinning of the film. Hence, in contrast to parallel film flows, non-parallel film flow down a cone is only locally 'unstable' with respect to disturbances which grow in an absolute sense; that is, only certain regions of the cone are subject to amplified disturbances. Therefore film flow down a cone is globally asymptotically stable, but not uniformly asymptotically stable. Similar behaviour has been predicted by Eagles & Weissman (1975) for the linear stability of slowly varying flow in a diverging straight-walled channel.

It is hoped that the solution presented here is a significant first step in our understanding of non-parallel film flows which permits us to ascertain the principal effect of a decreasing local Reynolds number on the linear stability of this flow.

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